APPLICATION OF THE PELL EQUATION TO THE PROBLEM OF EXTRACTION OF THE CUBE ROOT OF A BINOMIAL SURD.

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In the solution of cubic equations and in certain other cases it is desirable to have a somewhat general method for extracting the cube root of a binomial surd. This may be accomplished by the aid of the Pell Equation. To illustrate by an example let it be required to find the cube root of

$$553090\sqrt{2} + 53848\sqrt{211}.$$

Let

$$(553090\sqrt{2} + 53848\sqrt{211})^{\frac{1}{3}} = m\sqrt{2} + n\sqrt{211}.$$

Then

$$(553090\sqrt{2} - 53848\sqrt{211})^{\frac{1}{3}} = m\sqrt{2} - n\sqrt{211}.$$

Multiplying these two equations,

$$2m^2 - 211n^2 = (611817096200 - 611817098944)^{\frac{1}{3}},$$
$$2m^2 - 211n^2 = (-2744)^{\frac{1}{3}},$$
$$2m^2 - 211n^2 = -14.$$

Let

$$n = 2y.$$

Then

$$m^2 - 422y^2 = -7$$

The solution of this generalized Pell equation may be obtained from one of the convergents of $\sqrt{422}$ when developed into a continued fraction,

$$\sqrt{422} = 20 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5 \cdots}}}$$

or written in another form*

* For tables of continued fractions and for various methods of solving Pell equations see "The Pell Equation," E. E. Whitford, College of the City of New York, 1912.

$$\sqrt{422} = 20,\ 1,\ 1,\ 5,\ 2,\ 1,\ 3,\ (20)\ \cdots$$
$$1, 22, 19, 7, 14, 23, 11, (2)\ \cdots,$$

where the numbers in the second line denote the values of the form $m^2 - 422y^2$; and hence tell which convergent to select, in this case the fourth. The values of the form are alternately $+$ and $-$. The convergents are found to be

$$\frac{1}{0},\ \frac{20}{1},\ \frac{21}{1},\ \frac{41}{1}.$$

We select the fourth since for this the value of the form is -7,

$$m = 41.\quad y = 2,$$
$$n = 4.$$

then

$$\therefore (553090\sqrt{2} + 53848\sqrt{211})^{\frac{1}{3}} = 41\sqrt{2} + 4\sqrt{211}.$$

In solving $m^2 - Cy^2 = H$, if $H > 2\sqrt{C}$ certain modifications of the method have to be introduced, but where the solution of the cube root of a binomial surd exists in the form of $m\sqrt{p} \pm n\sqrt{q}$, m, n, p, q, positive integers, this method is always theoretically possible.

Second Illustration:

Extract the cube root of

$$25762\sqrt{2} + 14260\sqrt{7}.$$

(1) Let

$$(25762\sqrt{2} + 14260\sqrt{7}) = m\sqrt{2} + n\sqrt{7};$$

(2) then

$$(25762\sqrt{2} - 14260\sqrt{7})^{\frac{1}{3}} = m\sqrt{2} - n\sqrt{7}.$$

Multiplying (1) and (2)

$$(1327361288 - 1423433200)^{\frac{1}{3}} = 2m^2 - 7n^2;$$
$$\therefore 2m^2 - 7n^2 = (-96071912)^{\frac{1}{3}},$$
$$2m^2 - 7n^2 = -458.$$

Let $n = 2q$.

$$2m^2 - 28q^2 = -458,$$

(3) $$m^2 - 14q^2 = -229.$$

Now in seeking to solve the equation

(A) $$m^2 - Cq^2 = H, \text{ if } H > 2\sqrt{C},$$

the roots, m, q, can be found from the roots of a similar equation

$$z^2 - Cy^2 = H_1,$$

by the formulas

$$(B) \qquad m = \frac{K_1 z + Cy}{H_1}, \quad q = \frac{K_1 y + z}{H_1},$$

provided $K_1 < \frac{1}{2}H$, and such that

$$\frac{K_1^2 - C}{H} = H_1, \text{ an integer.}$$

For substituting the values in (B) into equation (A)

$$\frac{K_1^2 z^2 + 2K_1 Czy + C^2 y^2}{H_1^2} - \frac{CK_1^2 y^2 + 2CK_1 yz + Cz^2}{H_1^2} = H,$$

$$\left(\frac{K_1^2 - C}{H}\right) z^2 - C\left(\frac{K_1^2 - C}{H}\right) y^2 = H_1^2,$$

$$z^2 - Cy^2 = H_1.$$

The roots, z, y, can in their turn be found from the roots of a similar equation

$$l^2 - Ck^2 = H_2,$$

and the process can be repeated until the H_t is less than $2\sqrt{C}$.

Pursuing this method in solving equation (3) we seek

$$K_1 < \tfrac{1}{2} \cdot 229$$

and such that

$$\frac{K_1^2 - 14}{-229} = H_1$$

and find

$$\frac{2304 - 14}{-229} = -10;$$

$$\therefore K_1 = 48, \quad H_1 = -10.$$

Now applying this method to

$$z^2 - 14y^2 = -10,$$

seeking $K_2 < \frac{1}{2} \cdot 10$, and such that

$$\frac{K_2^2 - 14}{-10} = H_2,$$

we find

$$\frac{4 - 14}{-10} = 1;$$

$$\therefore K_2 = 2, \quad H_2 = 1.$$

But a solution of

$$l^2 - 14k^2 = 1,$$

is evidently $l = 1, \quad k = 0$.

Then by the formulas

$$z = \frac{K_2 l + Ck}{H_2}, \quad y = \frac{K_2 k + l}{H_2};$$

$$z = \frac{2 \cdot 1 + 14 \cdot 0}{1} = 2, \quad y = \frac{2 \cdot 0 + 1}{1} = 1,$$

and by the formulas (B)

$$m = \frac{48 \cdot 2 + 14 \cdot 1}{-10}, \quad q = \frac{48 \cdot 1 + 2}{-10},$$

$$m = 11, \quad q = 5,$$

disregarding quality sign, and since $n = 2q$, $n = 10$,

$$\therefore (25762\sqrt{2} + 14260\sqrt{7})^{\frac{1}{3}} = 11\sqrt{2} + 10\sqrt{7}.$$

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